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CONVOLUTION RELATIONS FOR SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract.

The purpose of this paper is to establish certain results concerning convolution relations for the subclass $T_n(\sim, \ ,¥)$ of univalent functions with negative coefficients.

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Introduction:

Let $H(U)$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the unit disc $U = \{ z : |z| < 1 \}$.Furthermore, let S denote the subclass of

H, which consists of functions of the form (1.1) that are univalent in U and satisfy conditions of normalization $f(0) = 0,\ f'(0) = 1$. A function of the form (1.1) which is member of S, is said to be starlike of order $\sim\ (0 \le \sim < 1)$, if and only if $\mathrm{Re}(zf'(z)/f(z)) > \sim,\ z \in U$, and is said to be convex of order $\sim\ (0 \le \sim < 1)$, if and only if $\mathrm{Re}(1 + zf''(z)/f'(z)) > \sim,\ z \in U$. We denote these classes respectively by $S^*(\sim)$ and $C(\sim)$. Let T denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed in the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \qquad a_j \ge 0,\ j = 2,3,\dots \ . \tag{1.2}$$

We denote N_0, the set of all non-negative integers (N_0 = (0,1,2,…)).

Definition 1 [3]. We define the operator $D^n : H(U) \to H(U)$, $n \in N_0$, by

$$D^0 f(z) = f(z) .$$
$$D^1 f(z) = zf\,'(z) .$$
$$D^n f(z) = D(D^{n-1} f(z)) .$$

Definition 2[1]. Let $\sim \in [\,0,1\,)$, $\in (\,0,1\,]$, $¥ \in (\,1/2,1\,]$, and let $n \in N_0$, we define, the class denoted $S_n(\sim, \ ,¥)$ by

$S_n(\sim, \ ,¥) = \{ f \in H(U) : f(0) = f\,'(0)-1 = 0$ and $\left| J_n(f,\sim,¥;z) \right| < \ , \ z \in U \}$
where

$$J_n(f,\sim,¥;z) = \frac{\left(\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)}{2¥\left(\dfrac{D^{n+1} f(z)}{D^n f(z)} - \sim \right) - \left(\dfrac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)} . \qquad (1.3)$$

We note that $S_0(\sim,1,1)$ is class of starlike functions of order $\sim$ and $S_1(\sim,1,1)$ is class of convex functions of order $\sim$ and $S_n(\sim,1,1)$ is class of n-starlike function of order $\sim$ defined in [3]; $S_n(\sim, \ ,1)$ is class of n-starlike function of order $\sim$ of type defined in [4]. We also note that the class $S_0(\sim, \ ,¥)$ is introduced and studied by Kulkarni [2].

Now,

$T_n(\sim, \ ,¥) = S_n(\sim, \ ,¥) \cap T$

We have established several general properties such as convolution, inclusion properties for aforementioned class $T_n(\sim, \ ,¥)$. In this paper we shall employ technique similar to those in [4] and [5].

Preliminary results

In [1] the next characterization of the class $T_n(\sim, \ ,¥)$ is given .

Theorem 1[1]. *Let* $\sim \in [\,0,1\,)$, $\in (\,0,1\,]$, $¥ \in (\,1/2,1\,]$, *and let* $n \in N_0$, *The function* f *of the form (1.2) is in* $T_n(\sim, \ ,¥)$ *if and only if*

$$\sum_{j=2}^{\infty} j^n \left[\, j-1+ \ (1-j+2¥\, j-2¥\sim) \,\right] a_j \le 2 \ ¥\,(1-\sim) \ . \qquad (2.1)$$

The result (2.1) is sharp, the extremal functions being given by

$$F_j(z) = z - \frac{2¥ \ (1-\sim)}{j^n\,[\, j-1+ \ (1-j+2¥\, j-2¥\sim)\,]}\, z^j \qquad j = 2, 3, \ldots . \qquad (2.2)$$

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Corollary 1. *If* $f \in T_n(\sim, \ \ ,¥)$ *then*

$$a_j \le \frac{2¥ \ \ (1-\sim)}{j^n [\, j-1+ \ \ (1-j+2¥ j-2¥\sim)\,]} \qquad j = 2, 3, \ldots$$

The result is sharp and the extremal functions are given by (2.2).

3. Convolution Properties

Let $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j$, $i = 1, 2, 3$. We define the Hadmard product or convolution of f_1 and f_2 by

$$\left(f_1 * f_2\right)(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j .$$

(3.1)

Theorem 2. *Let* $f_1, f_2 \in T_n(\sim, \ \ ,¥)$, $\sim \in [0,1)$, $\ \ \in (0,1]$, $¥ \in (1/2, 1]$, *and* $n \in N_0$, *then* $f_1 * f_2 \in T_n(\sim', \ \ ,¥)$, *where*

$$\sim' = \sim'\left(\sim, \ \ ,¥, n\right) = 1 - \frac{2 \ \ ¥ \left(1-\sim\right)^2 \left(1- \ \ +2 \ \ ¥\right)}{\left[1+ \ \ \left(-1+4¥ -2\sim¥\right)\right]^2 2^n - 4 \ \ ^2 ¥^2 \left(1-\sim\right)^2}$$

(3.2)

and $\sim < \sim'\left(\sim, \ \ ,¥, n\right) < 1$ *.The result is sharp, the extremal functions are* $f_1 = f_2 = F_2$, *where* F_2 *is given in Theorem* 1.

Proof. Let $f_1, f_2 \in T_n(\sim, \ \ ,¥)$, then from Theorem1 we have

$$\sum_{j=2}^{\infty} \frac{j^n \left[\, j-1+ \ \ (1-j+2¥ j-2¥\sim)\,\right]}{2 \ \ ¥ (1-\sim)} a_{1,j} \le 1 \qquad . \qquad (3.3)$$

and

$$\sum_{j=2}^{\infty} \frac{j^n \left[\, j-1+ \ \ (1-j+2¥ j-2¥\sim)\,\right]}{2 \ \ ¥ (1-\sim)} a_{2,j} \le 1 \qquad . \qquad (3.4)$$

From Theorem1 we also have $f_1 * f_2 \in T_n(\sim', \ \ ,¥)$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n \left[\, j-1+ \ \ (1-j+2¥ j-2¥\sim')\,\right]}{2 \ \ ¥ (1-\sim')} a_{1,j}\, a_{2,j} \le 1$$

(3.5)

We wish to determine the largest $\sim' = \sim'\left(\sim, \ \ ,¥, n\right)$ such that (3.4) holds.
From (3.3) and (3.4) we get by means of Cauchy-Schwarz inequality

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$$\sum_{j=2}^{\infty} \frac{j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]}{2 \quad ¥\,(1-\sim\,)} \sqrt{a_{1,j}\; a_{2,j}} \;\leq\; 1$$

(3.6)

which implies

$$\sqrt{a_{1,j}\; a_{2,j}} \;\leq\; \frac{2 \quad ¥\,(1-\sim\,)}{j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]}, \quad j \;=\; 2,3,\ldots \quad . \qquad (3.7)$$

We note that the next inequalities

$$\frac{j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,')\,\right]}{2 \quad ¥\,(1-\sim\,')}\; a_{1,j}\; a_{2,j} \;\leq$$

$$\leq\; \frac{j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]}{2 \quad ¥\,(1-\sim\,)} \sqrt{a_{1,j}\; a_{2,j}}, \quad j \;=\; 2,3,\ldots$$

(3.8)

imply (3.5). But the inequalities (3.8) are equivalent to

$$\frac{j-1+ \quad (1-j+2¥\, j-2¥\sim\,')}{(1-\sim\,')} \sqrt{a_{1,j}\; a_{2,j}} \;\leq$$

(3.9)

$$\leq\; \frac{j-1+ \quad (1-j+2¥\, j-2¥\sim\,)}{(1-\sim\,)} \quad ,\; j \;=\; 2,3,\ldots \quad .$$

By using (3.7) we have

$$\frac{j-1+ \quad (1-j+2¥\, j-2¥\sim\,')}{(1-\sim\,')} \sqrt{a_{1,j}\; a_{2,j}} \;\leq$$

$$\leq\; \frac{2 \quad ¥\,(1-\sim\,)\left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,')\,\right]}{(1-\sim\,')\; j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]} \;,\quad j \;=\; 2,3,\ldots \;\; .$$

In order to obtain (3.9) it will sufficient to show that

$$\frac{2 \quad ¥\,(1-\sim\,)\left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,')\,\right]}{(1-\sim\,')\; j^n \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]} \;\leq\; \frac{j-1+ \quad (1-j+2¥\, j-2¥\sim\,)}{(1-\sim\,)},$$

$j \;=\; 2,3,\ldots$

that is

$$2 \quad ¥\,(1-\sim\,)^2 \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,')\,\right] \leq$$

(3.10)

$$\leq\; (1-\sim\,')\; \left[\, j-1+ \quad (1-j+2¥\, j-2¥\sim\,)\,\right]^2 \; j^n, \quad j \;=\; 2,3,\ldots \quad ,$$

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The inequalities (3.10) are equivalent to $A\sim' \le B$, where

$$A = -4 \ \ {}^{2}¥^{2}\left(1-\sim\right)^{2} + \left[\ j-1+ \ \ (1-j+2¥ j-2¥\sim)\ \right]^{2} j^{n} > 0$$

and

$$B=\left[j-1+ \ \ (1-j+2¥ j-2¥\sim)\right]^{2} j^{n} - 2 \ ¥\left(1-\sim\right)^{2}\left(j-1\right) - 2 \ \ {}^{2}¥\left(1-\sim\right)^{2}\left(1-j+2¥ j\right),$$
$$j = 2,3,\ldots \ .$$

We obtain

$$\sim' \le \frac{B}{A} = 1 - \frac{2 \ ¥\left(1-\sim\right)^{2}\left(j-1\right)\left(1- \ \ +2 \ ¥\right)}{\left[\ j-1+ \ \ (1-j+2¥ j-2¥\sim)\ \right]^{2} j^{n} - 4 \ \ {}^{2}¥^{2}\left(1-\sim\right)^{2}} = c(j).$$

We have, $c(2) \le c(j)$, $j = 2,3,\ldots$ and we choose $\sim' = \sim'\left(\sim, \ \ ,¥,n\right) = c(2)$

and $\sim < \sim'$, because

$$\sim'\left(\sim, \ \ ,¥,n\right) - \sim \ \ge \ \sim'\left(\sim, \ \ ,¥,0\right) - \sim =$$

$$= \left[\frac{\left(1-\sim\right)\left(1- \ \ \right)\left(1- \ \ +2 \ ¥\left(3-\sim\right)\right)+4 \ \ {}^{2}¥^{2}\left(1-\sim\right)\left(2-\sim\right)}{\left(1- \ \ \right)\left(1- \ \ +2 \ ¥\left(3-\sim\right)\right)+4 \ \ {}^{2}¥^{2}\left(2-\sim\right)+2 \ ¥\left(1-\sim\right)\left(1- \ \ +2 \ ¥\right)}\right] > 0$$

,
and

$$1-\sim'\left(\sim, \ \ ,¥,n\right) = \frac{2¥ \ \ (1-\sim)^{2}(1- \ \ +2 \ ¥)}{\left[1+ \ \ \left(-1+4¥-2\sim¥\right)\right]^{2} 2^{n} - \left(2 \ \ ¥ \ \left(1-\sim\right)\right)^{2}} > 0.$$

The extremal functions are $f_1 = f_2 = F_2$. Indeed $\left(F_2 * F_2\right)(z) = z - c_2 z^2 \in T_n(\sim', \ \ ,¥)$, where

$$c_2 = \frac{2^{2-2n} \ \ {}^{2}¥^{2}\left(1-\sim\right)^{2}}{\left[1+ \ \ \left(-1+4¥-2\sim¥\right)\right]^{2}} \quad \text{and} \quad \sim' = \sim'\left(\sim, \ \ ,¥,n\right) \quad \text{because}$$

$$\sum_{j=2}^{\infty} \frac{j^{n}\left[\ j-1+ \ \ (1-j+2¥ j-2¥\sim')\right]}{2 \ ¥\ (1-\sim')} c_j =$$

$$= \frac{2^{n}\left[1+ \ \ (-1+4¥-2¥\sim')\right]}{2 \ ¥\ (1-\sim')} \frac{2^{2-2n} \ \ {}^{2}¥^{2}\left(1-\sim\right)^{2}}{\left[1+ \ \ \left(-1+4¥-2\sim¥\right)\right]^{2}} = 1 \ .$$

Corollary 2. *If* $f_1, f_2 \in T_n(\sim, \ \ ,¥)$, *then* $f_1 * f_2 \in T_n(\sim, \ \ ,¥)$.

Corollary 3. *If* $f_1, f_2 \in T_n(\sim, \ \ ,¥)$, *then* $f_1 * f_2 \in T_n(\ °,1,1)$, *where*

$$° = °(\sim, \ ,¥) = 1 - \frac{(1-\sim)^2 \ {}^2 ¥^2}{2^{n-2}\left[1+ \ (-1+4¥-2\sim¥)\right]^2 - {}^2(1-\sim)^2} .$$

Proof. A simple computation yields

$$|J_n(f,\sim,¥;z)| < \qquad \text{implies} \qquad \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| \le \frac{2 \ ¥(1-\sim)}{1+2 \ ¥ -} .$$

Therefore

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > ±(\sim, \ ,¥) ,$$

where

$$±(\sim, \ ,¥) = \frac{1- \ +2\sim \ ¥}{1- \ +2 \ ¥} .$$

Let $f_1, f_2 \in T_n(\sim, \ ,¥)$ then $f_1 * f_2 \in T_n(\sim', \ ,¥)$ and
$±(\sim', \ ,¥) = ±(\sim'(\sim, \ ,¥,n), \ ,¥) = °(\sim, \ ,¥,n)$.
Hence
$f_1 * f_2 \in T_n(\ ±(\sim', \ ,¥), 1, 1 \ ) = T_n(\ °(\sim, \ ,¥), 1, 1 \ )$.

Theorem 3. *If* $f_1 \in T_n(\sim, \ ,¥)$ *and* $f_2 \in T_n(\sim', \ ,¥)$, *then* $f_1 * f_2 \in T_n(\sim'', \ ,¥)$, *where*

$$\sim''(\sim,\sim', \ ,¥,n) = 1 - \frac{2 \ ¥(1-\sim)(1-\sim')(1- \ +2 \ ¥)}{2^n\left[1+ \ (-1+4¥-2\sim¥)\right]\left[1+ \ (-1+4¥-2\sim'¥)\right] - 4 \ {}^2¥^2(1-\sim)(1-\sim')}$$

(3.11)
The result is sharp.
Proof. Proceeding as in proof of Theorem 2, we obtain

$$\sim'' \le c(j) = 1 - \frac{2 \ ¥(1-\sim)(1-\sim')(j-1)(1- \ +2 \ ¥)}{j^n\left[j-1+ \ (1-j+2¥j-2¥\sim)\right]\left[j-1+ \ (1-j+2¥j-2¥\sim')\right] - 4 \ {}^2¥^2(1-\sim)(1-\sim')}$$

(3.12)

Since, $c(2) \le c(j)$, $j = 2,3,...,$ we choose $\sim'' = \sim''(\sim, \ ,¥,n) = c(2)$
which prove Theorem 3.
Finally by taking the functions

$$f_1(z) = z - \frac{2¥ \ (1-\sim)}{j^n[j-1+ \ (1-j+2¥j-2¥\sim)]} z^j \qquad \text{and}$$

$$f_2(z) = z - \frac{2¥ \ (1-\sim')}{j^n[j-1+ \ (1-j+2¥j-2¥\sim')]} z^j$$

we can see that the result is sharp.

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Corollary 4. *If* $f_i \in T_n(\sim, \ ,¥), \{i=1,2,3\}$, *then* $f_1 * f_2 * f_3 \in T_n(\sim'', \ ,¥)$, *where*

$$\sim''(\sim, \ ,¥,n)=1-\frac{2 \ {}^2¥^2(1-\sim)^3(1- \ +2 \ ¥)}{2^{2n}\left[1+ \ (-1+4¥-2\sim¥)\right]^3-8 \ {}^3¥^3(1-\sim)^3} .$$

(3.13)

Proof. From Theorem 2 we have $f_1 * f_2 \in T_n(\sim', \ ,¥)$, where $\sim'$ is given by (3.2). Then by
using Theorem 3 we get $f_1 * f_2 * f_3 \in T_n(\sim'', \ ,¥)$, where

$$\sim''(\sim, \ ,¥,n)=1-\frac{2 \ ¥(1-\sim)(1-\sim')(1- \ +2 \ ¥)}{2^n\left[1+ \ (-1+4¥-2\sim¥)\right]\left[1+ \ (-1+4¥-2\sim'¥)\right]-4 \ {}^2¥^2(1-\sim)(1-\sim')}$$

$$=1-\frac{2 \ {}^2¥^2(1-\sim)^3(1- \ +2 \ ¥)}{2^{2n}\left[1+ \ (-1+4¥-2\sim¥)\right]^3-8 \ {}^3¥^3(1-\sim)^3} .$$

Thus we have completed the proof of corollary4.

Theorem 4. *Let* $f_1, f_2 \in T_n(\sim, \ ,¥)$, $\sim \in [0,1)$, $\ \in (0,1]$, $¥ \in (1/2,1]$ *and* $n \in N_0$, *then* $f_1 * f_2 \in T_n(\sim, \ ',¥)$, *where*

$$' = \ '(\sim, \ ,¥,n) = \frac{2 \ {}^2¥(1-\sim)}{\left[1+ \ (-1+4¥-2\sim¥)\right]^2 2^n - 2 \ {}^2¥(1-\sim)(-1+4¥-2\sim¥)}$$

(3.14)

and $0 < \ '(\sim, \ ,¥,n) < \$ *.The result is sharp, the extremal functions are* $f_1 = f_2 = F_2$,
where F_2 *is given in Theorem 1.*

Proof. If $f_1, f_2 \in T_n(\sim, \ ,¥)$, then (3.3) and (3.4) hold. By Theorem 1 we have $f_1 * f_2 \in T_n(\sim, \ ',¥)$ if and only if

$$\sum_{j=2}^{\infty}\frac{j^n\left[j-1+ \ '(1-j+2¥j-2¥\sim)\right]}{2 \ '¥(1-\sim)}a_{1,j}\,a_{2,j} \le 1 \qquad (3.15)$$

We wish to determine the smallest $' = \ '(\sim, \ ,¥,n)$ such that (3.15) holds.
We note that the next inequalities

$$\frac{j-1+ \ '(1-j+2¥j-2¥\sim)}{'}\sqrt{a_{1,j}\,a_{2,j}} \le$$

(3.16)

$$\le \frac{j-1+ \ (1-j+2¥j-2¥\sim)}{} \ , \ j=2,3,... \ .$$

imply (3.15).

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By (3.7) we obtain

$$\frac{j-1+\ \ '(1-j+2¥\,j-2¥\sim)}{'}\sqrt{a_{1,j}\,a_{2,j}}\ \le$$

$$\le\ \frac{2\ \ ¥\,(1-\sim)\left[\,j-1+\ \ '(1-j+2¥\,j-2¥\sim)\,\right]}{'\ \ j^{n}\left[\,j-1+\ \ \ (1-j+2¥\,j-2¥\sim)\,\right]}\ ,\quad j\ =\ 2,3,\ldots\ .$$

In order to obtain (3.16) it will sufficient to show that

$$\frac{2\ \ ¥\,(1-\sim)\left[\,j-1+\ \ '(1-j+2¥\,j-2¥\sim)\,\right]}{'\ \ j^{n}\left[\,j-1+\ \ \ (1-j+2¥\,j-2¥\sim)\,\right]}\ \le$$

$$\le\ \frac{j-1+\ \ \ (1-j+2¥\,j-2¥\sim)}{\ },\quad j\ =\ 2,3,\ldots$$

These last inequalities are equivalent to

$$'\ \ge\ \frac{2\ \ ^{2}¥\,(1-\sim)\,(\,j-1\,)}{\left[\,1-j+\ \ \ (-1+4¥-2\sim¥)\,\right]^{2}j^{n}\ -\ 2\ \ \ ^{2}¥\,(1-\sim)(1-j+2¥\,j-2\sim¥)}\ =\ d(\,j\,)$$

(3.17)

We choose $\ '\ =\ \ '(\sim,\ \ ,¥,n)\ =\ d(2)$, because $d(2)\ \ge\ d(\,j\,)\ ,\ j\ =\ 2,3,\ldots\ .$

We have $\ '\ <\ \ \ $, because

$$-\ \ '(\sim,\ \ ,¥,n)\ \ge\ \ -\ \ '(\sim,\ \ ,¥,0)\ =$$

$$=\ \ \left[\frac{(1-\ \ )(1-\ \ +2\ ¥(3-\sim))+4\ \ ^{2}¥^{2}(2-\sim)}{(1-\ \ )(1-\ \ +2\ ¥(3-\sim))+4\ \ ^{2}¥^{2}(2-\sim)+2\ ¥(1-\sim)}\right]>\ 0$$

and $\ '(\sim,\ \ ,¥,n)\ >\ 0$ because

$$\left[\,1+\ \ \ (-1+4¥-2\sim¥)\,\right]^{2}2^{n}\ -\ 2\ \ \ ^{2}¥\,(1-\sim)(-1+4¥-2\sim¥)\ \ge$$

$$\ge\left[\,1+\ \ \ (-1+4¥-2\sim¥)\,\right]^{2}-\ 2\ \ \ ^{2}¥\,(1-\sim)(-1+4¥-2\sim¥)\ =$$

$$=(1-\ \ )\,(1-\ \ +2\ ¥(3-\sim))+4\ \ ^{2}¥^{2}(2-\sim)+2\ ¥(1-\sim)\ >\ 0$$

The extremal functions are $f_1\ =\ f_2\ =\ F_2$. Indeed $(F_2*F_2)(z)\ =\ z\ -\ c_2\,z^2\ \in\ T_n(\sim,\ \ ',¥)$,
where c_2 is given in the proof of Theorem 2 and $\ '\ =\ \ '(\sim,\ \ ,¥,n)$ because

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$$\sum_{j=2}^{\infty} \frac{j^n\left[\, j-1+ \quad '(1-j+2\yen\, j-2\yen\sim\,)\,\right]}{2 \quad '\yen\,(1-\sim\,)}\; c_j \;=\; \frac{2^n\left[\,1+ \quad '(-1+2\yen\, j-2\yen\sim\,)\,\right]}{2 \quad '\yen\,(1-\sim\,)}\; c_2 \;=\; 1.$$

Theorem 5. *If* $f_1 \in T_n(\sim, \quad ,\yen\,)$ *and* $f_2 \in T_n(\sim, \quad ',\yen\,)$, *then* $f_1 * f_2 \in T_n(\sim, \quad '',\yen\,)$, *where*

$$''(\sim, \quad , \quad ',\yen, n) = \frac{2 \quad '\yen\,(1-\sim\,)}{\left[1+ \quad (-1+4\yen-2\sim\yen\,)\right]\left[1+ \quad '(-1+4\yen-2\sim\yen\,)\right]2^n - 2 \quad '\yen\,(1-\sim\,)(-1+4\yen-2\sim\yen\,)}$$

(3.18)

The result is sharp.

Proof. Proceeding as in proof of Theorem4, we obtain

$$'' \ge d(j) = \frac{2 \quad '\yen\,(1-\sim\,)(\,j-1\,)}{\left[1-j+ \quad (-1+4\yen-2\sim\yen\,)\right]\left[1-j+ \quad '(-1+4\yen-2\sim\yen\,)\right]j^n - 2 \quad '\yen\,(1-\sim\,)(1-j+2\yen\, j-2\sim\yen\,)}$$

(3.19)

Since $d(2) \ge d(j)$, $j = 2, 3, \ldots$. we choose $'' = ''(\sim, \quad ,\yen, n) = d(2)$

which prove Theorem 5.

Finally by taking the functions

$$f_1(z) = z - \frac{2\yen \quad (1-\sim\,)}{j^n[\,j-1+ \quad (1-j+2\yen\, j-2\yen\sim)]}\, z^j$$

and

$$f_2(z) = z - \frac{2\yen \quad '(1-\sim\,)}{j^n[\,j-1+ \quad '(1-j+2\yen\, j-2\yen\sim)]}\, z^j$$

we can see that the result is sharp.

Corollary 5. *If* $f_i \in T_n(\sim, \quad ,\yen\,), \{i=1,2,3\}$, *then* $f_1 * f_2 * f_3 \in T_n(\sim, \quad '',\yen\,)$, *where*

$$''(\sim, \quad ,\yen, n) = \frac{2 \quad {}^3\yen^2\,(1-\sim\,)^2}{2^{2n}\left[1+ \quad (-1+4\yen-2\sim\yen\,)\right]^3 - 4 \quad {}^3\yen^2(1-\sim)^2(-1+4\yen-2\sim\yen\,)}\;.$$

(3.20)

Theorem 6. *Let* $f_1, f_2 \in T_n(\sim, \quad ,\yen\,)$, $\sim\, \in [\,0,1\,)$, $\quad \in (\,0,1\,]$, $\yen \in (\,1/2,1\,]$ *and* $n \in N_0$, *then* $f_1 * f_2 \in T_n(\sim, \quad ,\yen\,')$, *where*

$$\yen\,' = \yen\,'(\,\sim, \quad ,\yen, n\,) = \frac{2 \quad \yen^2\,(\,1-\sim\,)\,(\,1- \quad)}{\left[\,1+ \quad (\,-1+4\yen-2\sim\yen\,)\,\right]^2 2^n - 4 \quad {}^2\yen^2\,(\,1-\sim\,)(\,2-\sim\,)}$$

(3.21)

and $0 \le \yen\,'(\,\sim, \quad ,\yen, n\,) < \yen$ *.The result is sharp, the extremal functions are* $f_1 = f_2 = F_2$, *where* F_2 *is given in Theorem 1.*

Proof. If $f_1, f_2 \in T_n(\sim, \ ,\yen)$, then (3.3) and (3.4) hold. By Theorem 1 we have $f_1 * f_2 \in T_n(\sim, \ ,\yen')$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n\left[\, j-1+ \ \ (1-j+2\yen' j-2\yen'\sim)\,\right]}{2 \ \ \yen'(1-\sim)}\, a_{1,j}\, a_{2,j} \le 1 \qquad (3.22)$$

We wish to determine the smallest $\yen' = \yen'(\sim, \ ,\yen, n)$ such that (3.22) holds.
We note that the next inequalities imply (3.22).

$$\frac{j-1+ \ \ (1-j+2\yen' j-2\yen'\sim)}{\yen'}\sqrt{a_{1,j}\, a_{2,j}} \le$$

(3.23)

$$\le \frac{j-1+ \ \ (1-j+2\yen j-2\yen\sim)}{\yen}, \quad j = 2,3,\ldots \ .$$

By (3.7) we obtain

$$\frac{j-1+ \ \ (1-j+2\yen' j-2\yen'\sim)}{\yen'}\sqrt{a_{1,j}\, a_{2,j}} \le$$

$$\le \frac{2 \ \ \yen(1-\sim)\left[\, j-1+ \ \ (1-j+2\yen' j-2\yen'\sim)\,\right]}{\yen'\, j^n\left[\, j-1+ \ \ (1-j+2\yen j-2\yen\sim)\,\right]}, \quad j = 2,3,\ldots \ .$$

In order to obtain (3.23) it will sufficient to show that

$$\frac{2 \ \ \yen(1-\sim)\left[\, j-1+ \ \ (1-j+2\yen' j-2\yen'\sim)\,\right]}{\yen'\, j^n\left[\, j-1+ \ \ (1-j+2\yen j-2\yen\sim)\,\right]} \le$$

$$\le \frac{j-1+ \ \ (1-j+2\yen j-2\yen\sim)}{\yen}, \quad j = 2,3,\ldots$$

These last inequalities are equivalent to

$$\yen' \ge \frac{2 \ \ \yen^2(1-\sim)(1- \ \ )(j-1)}{\left[1-j+ \ \ (-1+4\yen-2\sim\yen)\right]^2 j^n - 4 \ \ ^2\yen^2(1-\sim)(j-\sim)} = e(j)$$

(3.24)

We choose $\yen' = \yen'(\sim, \ ,\yen, n) = e(2)$, because $e(2) \ge e(j)$, $j = 2,3,\ldots$.
We have $\yen' < \yen$, because

$$\yen - \yen'(\sim, \ ,\yen, n) \ge \yen - \yen'(\sim, \ ,\yen, 0) =$$

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$$= ¥\left[\frac{(1- \ )(1- \ +2 \ ¥(3-\sim \ ))+4 \ ^2¥^2(2-\sim \ )}{(1- \ )(1- \ +2 \ ¥(3-\sim \ ))+4 \ ^2¥^2(2-\sim \ )+2 \ ¥(1-\sim \ )(1- \ )}\right] > 0$$

and $¥'(\sim, \ ,¥,n) \geq 0$ because

$$\left[1+ \ (-1+4¥-2\sim¥ \ )\right]^2 2^n - 4 \ ^2¥^2(1-\sim \ )(2-\sim \ ) \geq$$

$$\geq \left[1+ \ (-1+4¥-2\sim¥ \ )\right]^2 - 4 \ ^2¥^2(1-\sim \ )(2-\sim \ ) =$$

$$= (1- \ )(1- \ +2 \ ¥(3-\sim \ ))+4 \ ^2¥^2(2-\sim \ )+2 \ ¥(1-\sim \ )(1- \ ) > 0$$

The extremal functions are $f_1 = f_2 = F_2$. Indeed $(F_2 * F_2)(z) = z - c_2 z^2 \in T_n(\sim, \ ,¥')$,
where c_2 is given in the proof of Theorem 2 and $¥' = ¥'(\sim, \ ,¥,n)$ because

$$\sum_{j=2}^{\infty} \frac{j^n\left[j-1+ \ (1-j+2¥'j-2¥'\sim \ )\right]}{2 \ ¥'(1-\sim \ )} c_j = \frac{2^n\left[1+ \ (-1+2¥'j-2¥'\sim \ )\right]}{2 \ ¥'(1-\sim \ )} c_2 = 1.$$

Theorem 7. *If* $f_1 \in T_n(\sim, \ ,¥)$ *and* $f_2 \in T_n(\sim, \ ,¥')$, *then* $f_1 * f_2 \in T_n(\sim, \ ,¥'')$, *where*

$$¥''(\sim, \ ,¥,n) = \frac{2 \ ¥¥'(1-\sim)(1- \ )}{2^n\left[-1+ \ (-1+4¥-2\sim¥)\right]\left[-1+ \ (-1+4¥'-2\sim¥)\right]-4 \ ^2¥¥'(1-\sim)(2-\sim)}$$

.
(3.25)
The result is sharp.

Proof. Proceeding as in proof of Theorem6, we obtain

$$¥' \geq e(j) = \frac{2 \ ¥¥'(1-\sim)(1- \ )(j-1)}{j^n\left[1-j+ \ (-1+4¥-2\sim¥)\right]\left[1-j+ \ (-1+4¥'-2\sim¥)\right]-4 \ ^2¥¥'(1-\sim)(j-\sim)}$$

(3.26)

Since $e(2) \geq e(j)$, $j = 2,3,\ldots$, we choose $¥' = ¥'(\sim, \ ,¥,n) = e(2)$
which prove Theorem 7.
Finally by taking the functions

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$$f_1(z) = z - \frac{2¥ \quad (1-\sim)}{j^n[j-1+ \quad (1-j+2¥ j-2¥\sim)]} z^j$$ and

$$f_2(z) = z - \frac{2¥' \quad (1-\sim)}{j^n[j-1+ \quad (1-j+2¥' j-2¥'\sim)]} z^j$$

we can see that the result is sharp.

Corollary 6. *If* $f_i \in T_n(\sim, \quad ,¥), \{i=1,2,3\}$, *then* $f_1 * f_2 * f_3 \in T_n(\sim, \quad ,¥'')$, *where*

$$¥''(\sim, \quad ,¥, n) = \frac{4 \quad {}^2¥^3 (1-\sim)^2 (1- \quad)}{2^{2n}\left[-1+ \quad (-1+4¥-2\sim¥)\right]^3 - 8 \quad {}^3¥^3 (1-\sim)^2 (2-\sim)}.$$

(3.27)

Remark. We can also obtain corollary 2 and corollary 3 by using Theorem 4 and Theorem 6.

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