

CONVOLUTION RELATIONS FOR SUBCLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract.

The purpose of this paper is to establish certain results concerning convolution relations for the subclass $T_n(r, s, \chi)$ of univalent functions with negative coefficients.

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Introduction:

Let $H(U)$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (1.1)$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Furthermore, let S denote the subclass of

H , which consists of functions of the form (1.1) that are univalent in U and satisfy conditions of normalization $f(0) = 0, f'(0) = 1$. A function of the form (1.1) which is member of S , is said to be starlike of order r ($0 \leq r < 1$), if and only if $\operatorname{Re}(zf'(z)/f(z)) > r, z \in U$, and is said to be convex of order r ($0 \leq r < 1$), if and only if $\operatorname{Re}(1 + zf''(z)/f'(z)) > r, z \in U$.

We denote these classes respectively by $S^*(r)$ and $C(r)$. Let T denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed in the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j \quad a_j \geq 0, j = 2, 3, \dots \quad (1.2)$$

We denote N_0 , the set of all non-negative integers ($N_0 = (0, 1, 2, \dots)$).

Definition 1 [3]. We define the operator $D^n : H(U) \rightarrow H(U)$, $n \in N_0$, by

$$\begin{aligned} D^0 f(z) &= f(z). \\ D^1 f(z) &= z f'(z). \\ D^n f(z) &= D(D^{n-1} f(z)). \end{aligned}$$

Definition 2 [1]. Let $\gamma \in [0, 1)$, $s \in (0, 1]$, $\alpha \in (1/2, 1]$, and let $n \in N_0$, we define, the class denoted $S_n(\gamma, s, \alpha)$ by

$S_n(\gamma, s, \alpha) = \{ f \in H(U) : f(0) = f'(0) - 1 = 0 \text{ and } |J_n(f, \gamma, \alpha; z)| < s, z \in U \}$
where

$$J_n(f, \gamma, \alpha; z) = \frac{\left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)}{2\alpha \left(\frac{D^{n+1} f(z)}{D^n f(z)} - \gamma \right) - \left(\frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right)}. \quad (1.3)$$

We note that $S_0(\gamma, 1, 1)$ is class of starlike functions of order γ and $S_1(\gamma, 1, 1)$ is class of convex functions of order γ and $S_n(\gamma, 1, 1)$ is class of n -starlike function of order γ defined in [3]; $S_n(\gamma, s, 1)$ is class of n -starlike function of order γ of type s defined in [4].

We also note that the class $S_0(\gamma, s, \alpha)$ is introduced and studied by Kulkarni [2].

Now,

$$T_n(\gamma, s, \alpha) = S_n(\gamma, s, \alpha) \cap T$$

We have established several general properties such as convolution, inclusion properties for aforementioned class $T_n(\gamma, s, \alpha)$. In this paper we shall employ technique similar to those in [4] and [5].

Preliminary results

In [1] the next characterization of the class $T_n(\gamma, s, \alpha)$ is given .

Theorem 1 [1]. Let $\gamma \in [0, 1)$, $s \in (0, 1]$, $\alpha \in (1/2, 1]$, and let $n \in N_0$, The function f of the form (1.2) is in $T_n(\gamma, s, \alpha)$ if and only if

$$\sum_{j=2}^{\infty} j^n [j-1 + s(1-j+2\alpha j-2\alpha\gamma)] a_j \leq 2s\alpha(1-\gamma). \quad (2.1)$$

The result (2.1) is sharp, the extremal functions being given by

$$F_j(z) = z - \frac{2s\alpha(1-\gamma)}{j^n [j-1 + s(1-j+2\alpha j-2\alpha\gamma)]} z^j \quad j = 2, 3, \dots \quad (2.2)$$

Corollary 1. If $f \in T_n(r, s, x)$ then

$$a_j \leq \frac{2sx(1-r)}{j^n [j-1+s(1-j+2xj-2xr)]} \quad j = 2, 3, \dots$$

The result is sharp and the extremal functions are given by (2.2).

3. Convolution Properties

Let $f_i(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j$, $i=1,2,3$. We define the Hadamard product or convolution of f_1

and f_2 by

$$(f_1 * f_2)(z) = z - \sum_{j=2}^{\infty} a_{i,j} z^j .$$

(3.1)

Theorem 2. Let $f_1, f_2 \in T_n(r, s, x)$, $r \in [0, 1)$, $s \in (0, 1]$, $x \in (1/2, 1]$, and $n \in N_0$, then $f_1 * f_2 \in T_n(r', s, x)$, where

$$r' = r'(r, s, x, n) = 1 - \frac{2sx(1-r)^2(1-s+2sx)}{[1+s(-1+4x-2rx)]^2 2^n - 4s^2x^2(1-r)^2}$$

(3.2)

and $r < r'(r, s, x, n) < 1$. The result is sharp, the extremal functions are $f_1 = f_2 = F_2$, where F_2 is given in Theorem 1.

Proof. Let $f_1, f_2 \in T_n(r, s, x)$, then from Theorem 1 we have

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2xj-2xr)]}{2sx(1-r)} a_{1,j} \leq 1 .$$

(3.3)

and

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2xj-2xr)]}{2sx(1-r)} a_{2,j} \leq 1 .$$

(3.4)

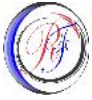
From Theorem 1 we also have $f_1 * f_2 \in T_n(r', s, x)$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2xj-2xr')] }{2sx(1-r')} a_{1,j} a_{2,j} \leq 1$$

(3.5)

We wish to determine the largest $r' = r'(r, s, x, n)$ such that (3.4) holds.

From (3.3) and (3.4) we get by means of Cauchy-Schwarz inequality



$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2xj-2xr)]}{2sx(1-r)} \sqrt{a_{1,j} a_{2,j}} \leq 1$$

(3.6)

which implies

$$\sqrt{a_{1,j} a_{2,j}} \leq \frac{2sx(1-r)}{j^n [j-1+s(1-j+2xj-2xr)]}, \quad j = 2, 3, \dots \quad (3.7)$$

We note that the next inequalities

$$\begin{aligned} & \frac{j^n [j-1+s(1-j+2xj-2xr')] }{2sx(1-r')} a_{1,j} a_{2,j} \leq \\ & \leq \frac{j^n [j-1+s(1-j+2xj-2xr)]}{2sx(1-r)} \sqrt{a_{1,j} a_{2,j}}, \quad j = 2, 3, \dots \end{aligned}$$

(3.8)

imply (3.5). But the inequalities (3.8) are equivalent to

$$\begin{aligned} & \frac{j-1+s(1-j+2xj-2xr')}{(1-r')} \sqrt{a_{1,j} a_{2,j}} \leq \\ & (3.9) \\ & \leq \frac{j-1+s(1-j+2xj-2xr)}{(1-r)}, \quad j = 2, 3, \dots \end{aligned}$$

By using (3.7) we have

$$\begin{aligned} & \frac{j-1+s(1-j+2xj-2xr')}{(1-r')} \sqrt{a_{1,j} a_{2,j}} \leq \\ & \leq \frac{2sx(1-r) [j-1+s(1-j+2xj-2xr')]}{(1-r') j^n [j-1+s(1-j+2xj-2xr)]}, \quad j = 2, 3, \dots \end{aligned}$$

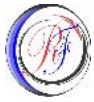
In order to obtain (3.9) it will sufficient to show that

$$\frac{2sx(1-r) [j-1+s(1-j+2xj-2xr')]}{(1-r') j^n [j-1+s(1-j+2xj-2xr)]} \leq \frac{j-1+s(1-j+2xj-2xr)}{(1-r)},$$

$j = 2, 3, \dots$

that is

$$\begin{aligned} & 2sx(1-r)^2 [j-1+s(1-j+2xj-2xr')] \leq \\ & (3.10) \\ & \leq (1-r') [j-1+s(1-j+2xj-2xr)]^2 j^n, \quad j = 2, 3, \dots \end{aligned}$$



The inequalities (3.10) are equivalent to $A r' \leq B$, where

$$A = -4s^2 x^2 (1-r)^2 + [j-1+s(1-j+2xj-2xr)]^2 j^n > 0$$

and

$$B = [j-1+s(1-j+2xj-2xr)]^2 j^n - 2sx(1-r)^2(j-1) - 2s^2x(1-r)^2(1-j+2xj),$$

$$j = 2, 3, \dots$$

We obtain

$$r' \leq \frac{B}{A} = 1 - \frac{2sx(1-r)^2(j-1)(1-s+2sx)}{[j-1+s(1-j+2xj-2xr)]^2 j^n - 4s^2x^2(1-r)^2} = c(j).$$

We have, $c(2) \leq c(j)$, $j = 2, 3, \dots$ and we choose $r' = r'(r, s, x, n) = c(2)$

and $r < r'$, because

$$r'(r, s, x, n) - r \geq r'(r, s, x, 0) - r =$$

$$= \left[\frac{(1-r)(1-s)(1-s+2sx(3-r)) + 4s^2x^2(1-r)(2-r)}{(1-s)(1-s+2sx(3-r)) + 4s^2x^2(2-r) + 2sx(1-r)(1-s+2sx)} \right] > 0$$

and

$$1-r'(r, s, x, n) = \frac{2xs(1-r)^2(1-s+2sx)}{[1+s(-1+4x-2rx)]^2 2^n - (2sx(1-r))^2} > 0.$$

The extremal functions are $f_1 = f_2 = F_2$. Indeed $(F_2 * F_2)(z) = z - c_2 z^2 \in T_n(r', s, x)$, where

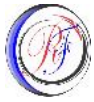
$$c_2 = \frac{2^{2-2n} s^2 x^2 (1-r)^2}{[1+s(-1+4x-2rx)]^2} \quad \text{and} \quad r' = r'(r, s, x, n) \quad \text{because}$$

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2xj-2xr')]^2}{2sx(1-r')} c_j =$$

$$= \frac{2^n [1+s(-1+4x-2rx')]^2}{2sx(1-r')} \frac{2^{2-2n} s^2 x^2 (1-r)^2}{[1+s(-1+4x-2rx)]^2} = 1.$$

Corollary 2. If $f_1, f_2 \in T_n(r, s, x)$, then $f_1 * f_2 \in T_n(r, s, x)$.

Corollary 3. If $f_1, f_2 \in T_n(r, s, x)$, then $f_1 * f_2 \in T_n(\dots, 1, 1)$, where



$$\dots = \dots(r, s, x) = 1 - \frac{(1-r)^2 s^2 x^2}{2^{n-2} [1+s(-1+4x-2rx)]^2 - s^2(1-r)^2}.$$

Proof. A simple computation yields

$$|J_n(f, r, x; z)| < s \quad \text{implies} \quad \left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right| \leq \frac{2sx(1-r)}{1+2sx-s}.$$

Therefore

$$\operatorname{Re} \left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \dagger(r, s, x),$$

where

$$\dagger(r, s, x) = \frac{1-s+2rsx}{1-s+2sx}.$$

Let $f_1, f_2 \in T_n(r, s, x)$ then $f_1 * f_2 \in T_n(r', s, x)$ and

$$\dagger(r', s, x) = \dagger(r'(r, s, x, n), s, x) = \dots(r, s, x, n).$$

Hence

$$f_1 * f_2 \in T_n(\dagger(r', s, x), 1, 1) = T_n(\dots(r, s, x), 1, 1).$$

Theorem 3. If $f_1 \in T_n(r, s, x)$ and $f_2 \in T_n(r', s, x)$, then $f_1 * f_2 \in T_n(r'', s, x)$, where

$$r''(r, r', s, x, n) = 1 - \frac{2sx(1-r)(1-r')(1-s+2sx)}{2^n [1+s(-1+4x-2rx)] [1+s(-1+4x-2r'x)] - 4s^2x^2(1-r)(1-r')}$$

(3.11)

The result is sharp.

Proof. Proceeding as in proof of Theorem 2, we obtain

$$r'' \leq c(j) = 1 - \frac{2sx(1-r)(1-r')(j-1)(1-s+2sx)}{j^n [j-1+s(1-j+2xj-2xr)] [j-1+s(1-j+2xj-2xr')] - 4s^2x^2(1-r)(1-r')}$$

(3.12)

Since, $c(2) \leq c(j)$, $j = 2, 3, \dots$, we choose $r'' = r''(r, s, x, n) = c(2)$

which prove Theorem 3.

Finally by taking the functions

$$f_1(z) = z - \frac{2xs(1-r)}{j^n [j-1+s(1-j+2xj-2xr)]} z^j \quad \text{and}$$

$$f_2(z) = z - \frac{2xs(1-r')}{j^n [j-1+s(1-j+2xj-2xr')] } z^j$$

we can see that the result is sharp.

Corollary 4. If $f_i \in T_n(r, s, x), \{i=1,2,3\}$, then $f_1 * f_2 * f_3 \in T_n(r'', s, x)$, where

$$r''(r, s, x, n) = 1 - \frac{2s^2x^2(1-r)^3(1-s+2sx)}{2^{2n} [1+s(-1+4x-2rx)]^3 - 8s^3x^3(1-r)^3} \tag{3.13}$$

Proof. From Theorem 2 we have $f_1 * f_2 \in T_n(r', s, x)$, where r' is given by (3.2). Then by

using Theorem 3 we get $f_1 * f_2 * f_3 \in T_n(r'', s, x)$, where

$$r''(r, s, x, n) = 1 - \frac{2sx(1-r)(1-r')(1-s+2sx)}{2^n [1+s(-1+4x-2rx)] [1+s(-1+4x-2r'x)] - 4s^2x^2(1-r)(1-r')}$$

$$= 1 - \frac{2s^2x^2(1-r)^3(1-s+2sx)}{2^{2n} [1+s(-1+4x-2rx)]^3 - 8s^3x^3(1-r)^3}$$

Thus we have completed the proof of corollary4.

Theorem 4. Let $f_1, f_2 \in T_n(r, s, x), r \in [0,1), s \in (0,1], x \in (1/2,1]$ and $n \in N_0$, then $f_1 * f_2 \in T_n(r, s', x)$, where

$$s' = s'(r, s, x, n) = \frac{2s^2x(1-r)}{[1+s(-1+4x-2rx)]^2 2^n - 2s^2x(1-r)(-1+4x-2rx)} \tag{3.14}$$

and $0 < s'(r, s, x, n) < s$. The result is sharp, the extremal functions are

$$f_1 = f_2 = F_2,$$

where F_2 is given in Theorem 1.

Proof. If $f_1, f_2 \in T_n(r, s, x)$, then (3.3) and (3.4) hold. By Theorem 1 we have

$f_1 * f_2 \in T_n(r, s', x)$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s'(1-j+2xj-2xr)]}{2s'x(1-r)} a_{1,j} a_{2,j} \leq 1 \tag{3.15}$$

We wish to determine the smallest $s' = s'(r, s, x, n)$ such that (3.15) holds.

We note that the next inequalities

$$\frac{j-1+s'(1-j+2xj-2xr)}{s'} \sqrt{a_{1,j} a_{2,j}} \leq$$

$$(3.16)$$

$$\leq \frac{j-1+s(1-j+2xj-2xr)}{s}, j = 2, 3, \dots$$

imply (3.15).

By (3.7) we obtain

$$\begin{aligned} & \frac{j-1+s'(1-j+2xj-2xr)}{s'} \sqrt{a_{1,j} a_{2,j}} \leq \\ & \leq \frac{2sx(1-r) [j-1+s'(1-j+2xj-2xr)]}{s' j^n [j-1+s(1-j+2xj-2xr)]}, \quad j = 2, 3, \dots \end{aligned}$$

In order to obtain (3.16) it will sufficient to show that

$$\begin{aligned} & \frac{2sx(1-r) [j-1+s'(1-j+2xj-2xr)]}{s' j^n [j-1+s(1-j+2xj-2xr)]} \leq \\ & \leq \frac{j-1+s(1-j+2xj-2xr)}{s}, \quad j = 2, 3, \dots \end{aligned}$$

These last inequalities are equivalent to

$$s' \geq \frac{2s^2x(1-r)(j-1)}{[1-j+s(-1+4x-2rx)]^2 j^n - 2s^2x(1-r)(1-j+2xj-2xr)} = d(j) \quad (3.17)$$

We choose $s' = s'(r, s, x, n) = d(2)$, because $d(2) \geq d(j)$, $j = 2, 3, \dots$.

We have $s' < s$, because

$$\begin{aligned} & s - s'(r, s, x, n) \geq s - s'(r, s, x, 0) = \\ & = s \left[\frac{(1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)}{(1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)+2sx(1-r)} \right] > 0 \end{aligned}$$

and $s'(r, s, x, n) > 0$ because

$$\begin{aligned} & [1+s(-1+4x-2rx)]^2 2^n - 2s^2x(1-r)(-1+4x-2rx) \geq \\ & \geq [1+s(-1+4x-2rx)]^2 - 2s^2x(1-r)(-1+4x-2rx) = \\ & = (1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)+2sx(1-r) > 0 \end{aligned}$$

The extremal functions are $f_1 = f_2 = F_2$. Indeed $(F_2 * F_2)(z) = z - c_2 z^2 \in T_n(r, s', x)$,

where c_2 is given in the proof of Theorem 2 and $s' = s'(r, s, x, n)$ because

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(-1+2xj-2xr)]}{2s'x(1-r)} c_j = \frac{2^n [1+s(-1+2xj-2xr)]}{2s'x(1-r)} c_2 = 1.$$

Theorem 5. If $f_1 \in T_n(r, s, x)$ and $f_2 \in T_n(r, s', x)$, then $f_1 * f_2 \in T_n(r, s'', x)$, where

$$s''(r, s, s', x, n) = \frac{2ss'x(1-r)}{[1+s(-1+4x-2rx)][1+s'(-1+4x-2rx)]2^n - 2ss'x(1-r)(-1+4x-2rx)} \tag{3.18}$$

The result is sharp.

Proof. Proceeding as in proof of Theorem 4, we obtain

$$s'' \geq d(j) = \frac{2ss'x(1-r)(j-1)}{[1-j+s(-1+4x-2rx)][1-j+s'(-1+4x-2rx)]j^n - 2ss'x(1-r)(1-j+2xj-2rx)} \tag{3.19}$$

Since $d(2) \geq d(j)$, $j = 2, 3, \dots$ we choose $s'' = s''(r, s, x, n) = d(2)$

which prove Theorem 5.

Finally by taking the functions

$$f_1(z) = z - \frac{2xs(1-r)}{j^n[j-1+s(1-j+2xj-2xr)]} z^j \quad \text{and}$$

$$f_2(z) = z - \frac{2xs'(1-r)}{j^n[j-1+s'(1-j+2xj-2xr)]} z^j$$

we can see that the result is sharp.

Corollary 5. If $f_i \in T_n(r, s, x)$, $\{i=1,2,3\}$, then $f_1 * f_2 * f_3 \in T_n(r, s'', x)$, where

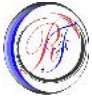
$$s''(r, s, x, n) = \frac{2s^3x^2(1-r)^2}{2^{2n}[1+s(-1+4x-2rx)]^3 - 4s^3x^2(1-r)^2(-1+4x-2rx)}. \tag{3.20}$$

Theorem 6. Let $f_1, f_2 \in T_n(r, s, x)$, $r \in [0, 1)$, $s \in (0, 1]$, $x \in (1/2, 1]$ and $n \in N_0$, then $f_1 * f_2 \in T_n(r, s, x')$, where

$$x' = x'(r, s, x, n) = \frac{2sx^2(1-r)(1-s)}{[1+s(-1+4x-2rx)]^2 2^n - 4s^2x^2(1-r)(2-r)} \tag{3.21}$$

and $0 \leq x'(r, s, x, n) < x$. The result is sharp, the extremal functions are

$f_1 = f_2 = F_2$, where F_2 is given in Theorem 1.



Proof. If $f_1, f_2 \in T_n(r, s, x)$, then (3.3) and (3.4) hold. By Theorem 1 we have $f_1 * f_2 \in T_n(r, s, x')$ if and only if

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2x'j-2x'r)]}{2sx'(1-r)} a_{1,j} a_{2,j} \leq 1 \quad (3.22)$$

We wish to determine the smallest $x' = x'(r, s, x, n)$ such that (3.22) holds.

We note that the next inequalities imply (3.22).

$$\begin{aligned} & \frac{j-1+s(1-j+2x'j-2x'r)}{x'} \sqrt{a_{1,j} a_{2,j}} \leq \\ & (3.23) \\ & \leq \frac{j-1+s(1-j+2xj-2xr)}{x}, \quad j = 2, 3, \dots \end{aligned}$$

By (3.7) we obtain

$$\begin{aligned} & \frac{j-1+s(1-j+2x'j-2x'r)}{x'} \sqrt{a_{1,j} a_{2,j}} \leq \\ & \leq \frac{2sx(1-r)[j-1+s(1-j+2x'j-2x'r)]}{x' j^n [j-1+s(1-j+2xj-2xr)]}, \quad j = 2, 3, \dots \end{aligned}$$

In order to obtain (3.23) it will sufficient to show that

$$\begin{aligned} & \frac{2sx(1-r)[j-1+s(1-j+2x'j-2x'r)]}{x' j^n [j-1+s(1-j+2xj-2xr)]} \leq \\ & \leq \frac{j-1+s(1-j+2xj-2xr)}{x}, \quad j = 2, 3, \dots \end{aligned}$$

These last inequalities are equivalent to

$$x' \geq \frac{2sx^2(1-r)(1-s)(j-1)}{[1-j+s(-1+4x-2rx)]^2 j^n - 4s^2x^2(1-r)(j-r)} = e(j) \quad (3.24)$$

We choose $x' = x'(r, s, x, n) = e(2)$, because $e(2) \geq e(j)$, $j = 2, 3, \dots$.

We have $x' < x$, because

$$x - x'(r, s, x, n) \geq x - x'(r, s, x, 0) =$$

$$= x \left[\frac{(1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)}{(1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)+2sx(1-r)(1-s)} \right] > 0$$

and $x'(r, s, x, n) \geq 0$ because

$$\left[1+s(-1+4x-2rx) \right]^2 2^n - 4s^2x^2(1-r)(2-r) \geq$$

$$\geq \left[1+s(-1+4x-2rx) \right]^2 - 4s^2x^2(1-r)(2-r) =$$

$$= (1-s)(1-s+2sx(3-r))+4s^2x^2(2-r)+2sx(1-r)(1-s) > 0$$

The extremal functions are $f_1 = f_2 = F_2$. Indeed $(F_2 * F_2)(z) = z - c_2 z^2 \in T_n(r, s, x')$,

where c_2 is given in the proof of Theorem 2 and $x' = x'(r, s, x, n)$ because

$$\sum_{j=2}^{\infty} \frac{j^n [j-1+s(1-j+2x'j-2x'r)]}{2sx'(1-r)} c_j = \frac{2^n [1+s(-1+2x'j-2x'r)]}{2sx'(1-r)} c_2 = 1.$$

Theorem 7. If $f_1 \in T_n(r, s, x)$ and $f_2 \in T_n(r, s, x')$, then $f_1 * f_2 \in T_n(r, s, x'')$, where

$$x''(r, s, x, n) = \frac{2sxx'(1-r)(1-s)}{2^n \left[-1+s(-1+4x-2rx) \right] \left[-1+s(-1+4x'-2rx) \right] - 4s^2xx'(1-r)(2-r)}$$

(3.25)

The result is sharp.

Proof. Proceeding as in proof of Theorem 6, we obtain

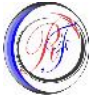
$$x' \geq e(j) = \frac{2sxx'(1-r)(1-s)(j-1)}{j^n \left[1-j+s(-1+4x-2rx) \right] \left[1-j+s(-1+4x'-2rx) \right] - 4s^2xx'(1-r)(j-r)}$$

(3.26)

Since $e(2) \geq e(j)$, $j = 2, 3, \dots$, we choose $x' = x'(r, s, x, n) = e(2)$

which prove Theorem 7.

Finally by taking the functions



$$f_1(z) = z - \frac{2xs(1-r)}{j^n[j-1+s(1-j+2xj-2xr)]} z^j \quad \text{and}$$

$$f_2(z) = z - \frac{2x's(1-r)}{j^n[j-1+s(1-j+2x'j-2x'r)]} z^j$$

we can see that the result is sharp.

Corollary 6. If $f_i \in T_n(r, s, x)$, $\{i=1,2,3\}$, then $f_1 * f_2 * f_3 \in T_n(r, s, x)$, where

$$x''(r, s, x, n) = \frac{4s^2x^3(1-r)^2(1-s)}{2^{2n}[-1+s(-1+4x-2rx)]^3 - 8s^3x^3(1-r)^2(2-r)}.$$

(3.27)

Remark. We can also obtain corollary 2 and corollary 3 by using Theorem 4 and Theorem 6.

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